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On sequences $\mathrm{cat}(f^k)$

Yu.B. Rudyak

Department of Mathematics, University of Florida, 358 Little Hall, P.O. Box 118105, Gainesville, FL 32611-8105, USA

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Dedicated to Jed Keesling with an occasion of his 60th birthday

Abstract

We prove that, for every decreasing sequence $\{a_k\}$ of natural numbers, there exists a map $f : X \to X$ with $\mathrm{cat}\, f^k = a_k$.

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Let $f : X \to Y$ be a map of finite CW-spaces. An *f-categorical set* is an open subset A of X such that $f|A : A \to Y$ is null-homotopic. An *f-categorical covering* is a covering $\{A_i\}$ of X such that every set A_i is f-categorical. The Lusternik–Schnirelmann category $\mathrm{cat}\, f$ of f is defined to be the minimal number k such that there exists a k-elemented f-categorical covering, [3,2,1]. We also set $\mathrm{cat}\, X = \mathrm{cat}\, 1_X$, cf. [4].

Given a pointed space, we always denote its base point by $*$.

Proposition 1 [1]. *For every diagram*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we have: $\mathrm{cat}(gf) \leqslant \min\{\mathrm{cat}\, f, \mathrm{cat}\, g\}$.

Proof. Clearly, every f-categorical covering is gf-categorical one. Furthermore, if $\{A_i\}$ is a g-categorical covering then $\{f^{-1}(A_i)\}$ is gf-categorical covering. □

E-mail address: rudyak@math.ufl.edu (Yu.B. Rudyak).

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Proposition 2 (cf. [3]). *For two maps $f : X \to Y$ and $g : A \to B$ we have*

$$\operatorname{cat}(f \vee g) = \max\{\operatorname{cat} f, \operatorname{cat} g\}$$

where $f \vee g : X \vee A \to Y \vee B$ is the wedge of f and g.

Proof. Let $\{U_1, \dots, U_k\}$ be an f-categorical covering and let $\{V_1, \dots, V_l\}$ be a g-categorical covering, $k \geqslant l$. The image of U_i in $X \vee A$ will also be denoted by U_i, and similarly for V_j. We can and shall assume that $* \notin U_i$ for $i > 1$ and $* \notin V_j$ for $j > 1$. Then, clearly, $U_i \cup V_i$, $1 < i \leqslant l$ is an $f \vee g$-categorical set. Furthermore, since

$$(X \vee A) \setminus (U_1 \cup V_1) = (X \setminus U_1) \cup (A \setminus V_1),$$

we conclude that $U_1 \cup V_1$ is open in $X \vee A$, and therefore $U_1 \cup V_1$ is $f \vee g$-categorical. Now,

$$U_1 \cup V_1, \dots, U_l \cup V_l,\ U_{l+1}, \dots, U_k$$

is an $f \vee g$-categorical covering. Thus, $\operatorname{cat}(f \vee g) \leqslant \max\{\operatorname{cat} f, \operatorname{cat} g\}$. The converse inequality is obvious. $\square$

Consider now a map $f : X \to X$. Because of what we said above, the sequence $\{\operatorname{cat}(f^k)\}$ is a decreasing sequence of natural numbers. Here we prove that, conversely, every such a sequence can be realized as the sequence of the form $\operatorname{cat} f^k$. In other words, the following theorem holds.

Theorem. *Let $\{a_k\}$ be a sequence of natural numbers with $a_k \geqslant a_{k+1}$. Then there exists a self map $f : X \to X$ of a* CW*-space such that* $\operatorname{cat}(f^k) = a_k$ *and* $\operatorname{cat} X = \operatorname{cat} f = a_1$.

Proof. Clearly, every sequence $\{a_k\}$ as above stabilizes, i.e., there exists m such that $a_k = a_{k+1}$ for $k \geqslant m$. We set

$$D(\{a_k\}) = \min\{m \mid a_k = a_{k+1} \text{ for } k \geqslant m\}$$

and prove the theorem by induction on $D(\{a_k\})$.

For $D = 1$ the result is obvious. Namely, we take a space X with $\operatorname{cat} X = a_1$ (for example, the real projective space of dimension $a_1 - 1$) and set $f = 1_X$.

Now, suppose that the theorem holds for $D = n$, i.e., that for every sequence $\{c_k\}$ with $D(\{c_k\}) = n$ there exists a map $f : X \to X$ with $\operatorname{cat}(f^k) = c_k$. Consider a sequence $\{a_k\}$ with $D(\{a_k\}) = n + 1$ and set $b_k = a_{k+1}$, $k = 1, 2, \dots$. Then $D(\{b_k\}) = n$, and so there exists a map $f : X \to X$ with $\operatorname{cat} X = b_1$ and $\operatorname{cat}(f^k) = b_k$.

Let Y be a pointed space with $\operatorname{cat} Y = a_1$, and let $\varepsilon : Y \to \{*\} \subset Y$ be the constant map. Consider the map

$$u : Y \vee Y \xrightarrow{1 \vee \varepsilon} Y \vee Y \xrightarrow{\pi} Y = \{*\} \vee Y \subset Y \vee Y$$

where $\pi = \pi_Y : Y \vee Y \to Y$ is the folding map. Clearly, $\operatorname{cat} u \leqslant \operatorname{cat} Y$. On the other hand, since the composition

$$Y = Y \vee \{*\} \subset Y \vee Y \xrightarrow{u} Y \vee Y \xrightarrow{\pi} Y$$

is the identity map, we conclude that, by Proposition 1, $\operatorname{cat} Y = \operatorname{cat} 1_Y \leqslant \operatorname{cat} u$. Thus, $\operatorname{cat} u = \operatorname{cat} Y = a_1$.

Consider the map

$$v : X \vee X \xrightarrow{1 \vee f} X \vee X \xrightarrow{\pi} X = \{*\} \vee X \subset X \vee X$$

and set $g = u \vee v : Y \vee Y \vee X \vee X \to Y \vee Y \vee X \vee X$. We claim that $\operatorname{cat}(g^k) = a_k$.

First, by Proposition 2, for $k = 1$ we have

$$\operatorname{cat} g = \max\{\operatorname{cat} u, \operatorname{cat} v\} = \max\{a_1, \operatorname{cat} v\} = a_1,$$

because $\operatorname{cat} v \leqslant \operatorname{cat}(X \vee X) = \operatorname{cat} X = b_1 \leqslant a_1$.

Furthermore, we have $\operatorname{cat} g^k = \max\{\operatorname{cat} u^k, \operatorname{cat} v^k\}$. But u^k is the constant map for $k > 1$, and hence $\operatorname{cat} g^k = \operatorname{cat} v^k$. So, it suffices to prove that $\operatorname{cat} v^k = \operatorname{cat} f^{k-1}$ for $k > 1$.

It is easy to see that v^k has the form

$$v^k : X \vee X \xrightarrow{f^{k-1} \vee f^k} X \vee X \xrightarrow{\pi} X = \{*\} \vee X \subset X \vee X.$$

So, by Propositions 1 and 2,

$$\operatorname{cat} v^k \leqslant \operatorname{cat}\bigl(f^{k-1} \vee f^k\bigr) = \operatorname{cat} f^{k-1}.$$

On the other hand, f^{k-1} can be decomposed as

$$f^{k-1} : X = X \vee \{*\} \subset X \vee X \xrightarrow{v^k} X \vee X \xrightarrow{\pi} X,$$

and so $\operatorname{cat} f^{k-1} \leqslant \operatorname{cat} v^k$. Thus, $\operatorname{cat} v^k = \operatorname{cat} f^{k-1}$.

Finally, $\operatorname{cat}(Y \vee Y \vee X \vee X) = \operatorname{cat} Y = a_1$ by Proposition 2. This completes the proof. □

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